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GROUPS COVERED BY AN INFINITE NUMBER OF ABELIAN SUBGROUPS

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If a group G is covered by κ Abelian subgroups (κ an infinite cardinal) then $|G:Z(G)| \le 2^\kappa$ and this estimate is sharp. This answers a question of Faber, Laver and McKenzie.

1. Introduction

Apropos of an investigation of the commuting graph associated with a group G, P. Erdős asked whether a group G covered by κ Abelian subgroups must have its centre of bounded index. In 1978, V. Faber, R. Laver and R. McKenzie [1] showed that

$$|G : Z(G)| \le 2^{2^{2^\kappa}}.$$

M. J. Tomkinson [2] improved this bound in 1985 showing that

$$|G : Z(G)| \le 2^{2^\kappa}.$$

Moreover, Tomkinson proved that $|G:Z(G)| \le 2^\kappa$ if we have an irredundant covering (none of the abelian subgroups of the covering can be omitted). Our result is the following:

Theorem 1.1. *Let G be a group and κ an infinite cardinal. If G can be covered by κ Abelian subgroups, then $|G:Z(G)| \le 2^\kappa$.*

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This result gives a complete solution of Problem 2 of Faber, Laver and McKenzie. It is noted in[1] that the direct sum of 2^κ finite non-commutative groups can be covered by κ abelian subgroups and the index of the centre equals 2^κ.

2. Proof of the theorem

Since every Abelian subgroup of group G is contained in a maximal Abelian subgroup, we may assume without loss of generality that G has a family $\mathcal{K}$ of maximal Abelian subgroups such that $|\mathcal{K}|=\kappa$ and $\bigcup\mathcal{K}=G$. Then we have $C_G(A)=A$ for all $A\in\mathcal{K}$ and $Z(G)=\bigcap\mathcal{K}$.

First we prove that it is enough to show that $|A:A\cap B|\leq 2^\kappa$ for all $A,B\in\mathcal{K}$.

Lemma 2.2. *If $|A:A\cap B|\leq 2^\kappa$ for all $A,B\in\mathcal{K}$, then $|G:Z(G)|\leq 2^\kappa$.*

Proof. We obtain $|A:Z(G)|=|A:\bigcap\mathcal{K}|\leq\prod_{B\in\mathcal{K}}|A:A\cap B|\leq(2^\kappa)^\kappa=2^\kappa$, and $|G:Z(G)|=|\bigcup\mathcal{K}:Z(G)|\leq\sum_{A\in\mathcal{K}}|A:Z(G)|\leq\kappa 2^\kappa=2^\kappa$. ∎

Now let us fix $A,B\in\mathcal{K}$, $A\neq B$. We define by transfinite recursion a sequence of elements $b_\alpha\in B$ $(\alpha<\nu)$ such that

$$C_A(b_\alpha)\cap C_A(\{b_\beta\mid\beta<\alpha\})\subsetneqq C_A(\{b_\beta\mid\beta<\alpha\})$$

provided the latter does not equal $A\cap B=C_A(B)$. We also choose elements $a_\alpha\in C_A(\{b_\beta\mid\beta<\alpha\})\setminus C_A(b_\alpha)$. For the length of this sequence we have the following.

Lemma 2.3. $|\nu|\leq 2^\kappa$.

Proof. For each $\alpha<\nu$ we define $\mathcal{K}_\alpha=\{C\in\mathcal{K}\mid\exists\beta:\alpha<\beta<\nu,\ a_\beta b_\alpha\in C\}$. We claim that these sets are pairwise different. Indeed, let $\alpha<\beta<\nu$ and $C\in\mathcal{K}$ such that $a_\beta b_\alpha\in C$. Then, by definition, $C\in\mathcal{K}_\alpha$. If γ satisfies $\beta<\gamma<\nu$, then $[a_\beta b_\alpha,a_\gamma b_\beta]=[a_\beta,b_\beta]\neq 1$, using $[a_\beta,a_\gamma]=1$, $[b_\alpha,b_\beta]=1$, $[b_\alpha,a_\beta]=1$. Thus $a_\gamma b_\beta\notin C$, since C is abelian, so $C\notin\mathcal{K}_\beta$. Now $|\nu|\leq|\mathcal{P}(\mathcal{K})|\leq 2^\kappa$ follows. ∎

Let $\hat{B}=\{b_\alpha\mid\alpha<\nu\}$ and let $\hat{A}$ be a transversal (set of left coset representatives) to $A\cap B$ in A. For each $C\in\mathcal{K}$ we define on $\hat{A}\times\hat{B}$ a matrix M_C with entries in $G\cup\{\emptyset\}$. First let $\hat{A}_C=\{a\in\hat{A}\,|\,a\hat{B}\cap C\neq\emptyset\}$, $\hat{B}_C=\{b\in\hat{B}\,|\,\hat{A}b\cap C\neq\emptyset\}$. Then we define for $a\in\hat{A}$, $b\in\hat{B}$

$$M_C(a,b)=\begin{cases}[b^{-1},a], & \text{if } a\in\hat{A}_C, b\in\hat{B}_C;\\ \emptyset, & \text{otherwise.}\end{cases}$$

Later we shall need the following simple lemma.

Lemma 2.4. *If a,a_1,b,b_1 are elements of G and $[a,a_1]=[b,b_1]=[a_1b,ab_1]=1$ then $[b^{-1},a]=[b_1^{-1},a_1]$.*

Proof. $b^{-1}a_1^{-1}b_1^{-1}a^{-1}a_1bab_1=1$ implies $a_1^{-1}b_1^{-1}a_1a^{-1}[a^{-1},a_1]ba=b_1^{-1}b[b,b_1^{-1}]$, thus $b_1a_1^{-1}b_1^{-1}a_1\cdot a^{-1}bab^{-1}=1$ and $[b^{-1},a]=[b_1^{-1},a_1]$. ∎

Lemma 2.5. *The number of different rows of M_C is at most $1+|\hat{B}|\leq 2^\kappa$.*

Proof. If $a\in\hat{A}\setminus\hat{A}_C$, then the row corresponding to a consists of $\emptyset$'s. Let now $a\in\hat{A}_C$ with $b_1\in\hat{B}$ such that $ab_1\in C$. Suppose that $a^*\in\hat{A}_C$ is such that $a^*b_1\in C$ as well. Then we claim that the rows corresponding to a and a^* coincide. If $b\in\hat{B}\setminus\hat{B}_C$ then $M_C(a,b)=\emptyset=M_C(a^*,b)$. If $b\in\hat{B}_C$ with $a_1b\in C$, $a_1\in\hat{A}$ then $[ab_1,a_1b]=1$. The previous lemma implies that $[b^{-1},a]=[b_1^{-1},a_1]$. Similarly, we obtain $[b^{-1},a^*]=[b_1^{-1},a_1]$, hence $[b^{-1},a^*]=[b^{-1},a]$, that is $M_C(a,b)=M_C(a^*,b)$, as we claimed. So the number of different rows of M_C is at most $1+|\hat{B}|$, indeed, and by Lemma 2.3, $|\hat{B}|\leq 2^\kappa$. ∎

Lemma 2.6. *For every pair $a,a^*\in\hat{A}$, $a\neq a^*$, there is a $C\in\mathcal{K}$ such that the rows of M_C corresponding to a and a^* are different.*

Proof. Suppose that $M_C(a,b)=M_C(a^*,b)$ for all $C\in\mathcal{K}$, $b\in\hat{B}$. Take an arbitrary $b\in\hat{B}$. Since $\bigcup\mathcal{K}=G$, there is a $C\in\mathcal{K}$ with $ab\in C$. Then $a\in\hat{A}_C$, $b\in\hat{B}_C$ and $M_C(a,b)=[b^{-1},a]$. Hence $M_C(a^*,b)\neq\emptyset$, so $M_C(a^*,b)=[b^{-1},a^*]=[b^{-1},a]$. That means $a^{-1}a^*\in C_A(b)$. This holds for all $b\in\hat{B}$, hence $a^{-1}a^*\in C_A(\hat{B})=A\cap B$. As $\hat{A}$ is a transversal to $A\cap B$ in A, we obtain $a=a^*$. ∎

For each $a\in\hat{A}$, there is a vector with coordinates $M_C(a)$ for $C\in\mathcal{K}$. No two of these vectors are identical because of Lemma 2.6, so their number is less than or equal to $(2^\kappa)^\kappa=2^\kappa$.

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References

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